

# Every graph is a cut locus

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January 14, 2013

**Abstract** We show that every connected graph can be realized as the cut locus of some point on some Riemannian surface  $S$  which, in some cases, has constant curvature. We study the stability of such realizations, and their generic behavior.  
**Math. Subj. Classification (2010):** 53C22, 05C62

## 1 Introduction

Unless explicitly stated otherwise, by a Riemannian manifold here we always mean a complete, compact and connected manifold without boundary. We shall work most of the time with surfaces (2-dimensional manifolds)  $S$ , and let  $M$  denote manifolds of arbitrary dimension  $d$ .

All graphs we consider in the following are finite, connected and may have loops and multiple edges. For the simplicity of our exposition, we see every graph  $G$  as a 1-dimensional simplicial complex. The *cyclic part* of  $G$  is the minimal (with respect to inclusion) subgraph  $G^{cp}$  of  $G$ , to which  $G$  is contractible; i.e., the minimal subgraph of  $G$  obtained by repeatedly contracting external edges, and for each vertex remaining of degree two (if any) merging its incident edges. A graph is called *cyclic* if it is equal to its cyclic part, and it is called of *constant order* if all its vertices have the same degree.

The notion of cut locus was introduced by H. Poincaré [27] in 1905, and gained since then an important place in global Riemannian geometry. The *cut locus*  $C(x)$  of the point  $x$  in the Riemannian manifold  $M$  is the set of all extremities (different from  $x$ ) of maximal (with respect to inclusion) *segments* (i.e., shortest geodesics) starting at  $x$ ; for basic properties and equivalent definitions refer, for example, to [22] or [29].

For Riemannian surfaces  $S$  is known that  $C(x)$ , if not a single point, is a local tree (i.e., each of its points  $z$  has a neighborhood  $V$  in  $S$  such that the component  $K_z(V)$  of  $z$  in  $C(x) \cap V$  is a tree), even a tree if  $S$  is homeomorphic to the sphere. A *tree* is a set  $T$  any two points of which can be joined by a unique Jordan arc included in  $T$ . The *degree* of a point  $y$  of a local tree is the number of components of  $K_y(V) \setminus \{y\}$  if  $V$  is chosen such that  $K_y(V)$  is a tree. A tree is *finite* if it has finitely many points of degree  $\geq 3$ , each of which has finite degree.

S. B. Myers [24] for  $d = 2$ , and M. Buchner [5] for general  $d$ , established that the cut locus of a real analytic Riemannian manifold of dimension  $d$  is homeomorphic to a finite  $(d - 1)$ -dimensional simplicial complex. For a class of Liouville manifolds, in particular for hyperellipsoids in the Euclidean space  $\mathbb{R}^d$ , the cut locus is reduced to a disc of dimension of most  $(d - 1)$ , see [13] and [14].

For Riemannian metrics of  $S$  non-analytic, cut loci may be quite large sets. J. Hebda [9] showed, for any  $C^\infty$  metric on  $S$ , that the Hausdorff 1-measure of any compact subset of the cut locus of any point is finite. Independently and using different techniques, J. Itoh [12] proved the same result under the weaker assumption of  $C^2$  metric. The differentiability of the metric cannot be lowered more; for example, the main result in [31] states that on most (in the sense of Baire category) convex surfaces (known to be of differentiability class  $\mathcal{C}^1 \setminus \mathcal{C}^2$ ), most points are extremities of any cut locus.

The problem of constructing a Riemannian metric with preassigned cut locus on a given manifold also received a certain interest. H. Gluck and D. Singer [8] constructed a Riemannian metric such that a non triangulable set, consisting of infinitely many arcs with a common extremity, becomes a cut locus. Another example of infinite length cut locus was provided by J. Hebda [10], while the case of a submanifold as preassigned cut locus was considered by L. Bérard-Bergery [3]. J. Itoh [11] showed that for any Morse function on a differentiable surface  $S$ , with only one critical point of index 0 and no saddle connection, there exists a Riemannian metric on  $S$  with respect to which  $C_f$ , the union of all unstable manifolds of critical points of  $f$  with positive index, becomes a cut locus. Independently but in the same direction as [11], M. Y. Park showed that, under some sufficient conditions, for any smoothly embedded, connected, finite cubic graph  $G$  in the surface  $S$ , there exists a Riemannian metric  $\alpha$  on  $S$  and a point  $x$  in  $S$  such that the cut locus of  $x$  with respect to  $\alpha$  is  $G$  [25], and that this cut locus is stable with respect to the metric [26]. All these results assume the manifold be given,

and search for a metric with respect to which some subset of the manifold becomes a cut locus.

A different approach was considered in [16], where the authors showed that any combinatorial type of finite tree can be realized as a cut locus on some, initially unknown, doubly covered convex polygon.

Our results here give this approach much more generality, by showing (see Theorem 2.6) that every connected graph can be realized as a cut locus; i.e., there exist a Riemannian surface  $S_G = (S_G, h)$  and a point  $x \in S_G$  such that  $C(x)$  is isometric to  $G$ . This is a partial converse to Myers' theorem mentioned above. If moreover  $G$  is cyclic of constant order, then it can be realized on a surface of constant curvature (Theorem 3.1). In the second part of this paper we show that –roughly speaking– stability is a generic property of cut locus realizations.

In a forthcoming paper [18] we are concerned about the orientability of the surfaces  $S_G$  realizing the graph  $G$  as a cut locus.

Employing the notion of *cut locus structure* [17], one can also regard our results as completing with additional information the surface case in the results of Buchner [4], [5], [6].

Recently, and from a viewpoint different from ours, cut loci and infinite graphs were studied by O. Baues and N. Peyerimhoff [1], [2], and by M. Keller [21], while in discrete group theory a similar notion, *dead-end depth*, was studied by S. Cleary and T. R. Riley [7], and by T. R. Riley and A. D. Warshall [28].

## 2 Every graph is a cut locus

Recall that a *segment between a point  $x$  and a closed set  $K$*  not containing  $x$  is a segment from  $x$  to a point in  $K$ , not longer than any other such segment; the *cut locus  $C(K)$  of the closed set  $K \subset S$*  is the set of all points  $y \in S$  such that there is a segment from  $y$  to  $K$  not extendable as a segment beyond  $y$ .

A graph is *metric* if each of its edges is endowed with a positive number, called *length*.

**Definition 2.1** *Let  $G$  be a graph. A  $G$ -strip is a topological surface  $P_G$  with boundary, such that:*

- (i) *the boundary of  $P_G$  is homeomorphic to a circle, and*
- (ii)  *$P_G$  contains (a graph isometric to)  $G$  and is contractible to  $G$ .*

A Riemannian  $G$ -strip is a  $G$ -strip  $P_G$  endowed with a Riemannian metric such that the cut locus of  $\text{bd}(P_G)$  in  $P_G$  is precisely  $G$ .

If the graph  $G$  is metric, we ask in addition that the induced metric on  $G$  by the metric of  $P_G$  coincides to the original metric of  $G$ .

Basic examples show that a topological surface with boundary is not contractible to each graph it contains.

**Definition 2.2** We say that a graph (or a metric graph)  $G$  can be realized as a cut locus if there exist a Riemannian surface  $S_G = (S_G, h)$  and a point  $x$  in  $S_G$  such that  $G$  is isometric to  $C(x)$ .

A. D. Weinstein (Proposition C in [32]) proved the following.

**Lemma 2.3** Let  $M$  be a  $d$ -dimensional Riemannian manifold and  $D$  an  $d$ -disc embedded in  $M$ . There exists a new metric on  $M$  agreeing with the original metric on a neighborhood of  $M \setminus (\text{interior of } D)$  such that, for some point  $p$  in  $D$ , the exponential mapping at  $p$  is a diffeomorphism of the unit disc about the origin in the tangent space at  $p$  to  $M$ , onto  $D$ .

**Proposition 2.4** The following statements are equivalent:

- i) the metric graph  $G$  can be realized as a cut locus;
- ii) there exists a  $G$ -strip;
- iii) there exists a Riemannian  $G$ -strip.

*Proof:* (i)  $\rightarrow$  (ii) Consider a point  $x$  on a Riemannian surface  $(S, g)$ , and a segment  $\gamma : [0, l_\gamma] \rightarrow S$  parametrized by arclength, with  $\gamma(0) = x$  and  $\gamma(l_\gamma) \in C(x)$ . For  $\varepsilon > 0$  strictly smaller than the injectivity radius  $\text{inj}(x)$  at  $x$ , the point  $\gamma(l_\gamma - \varepsilon)$  is well defined because  $\text{inj}(x) \leq l_\gamma$ . Since  $S \setminus C(x)$  is contractible to  $x$  along geodesic segments, and thus homeomorphic to an open disk, the union over all  $\gamma$ s of those points  $\gamma(l_\gamma - \varepsilon)$  is homeomorphic to the unit circle.

(ii)  $\rightarrow$  (iii) An explicit construction of a Riemannian  $G$ -strip from a given  $G$ -strip was provided by the first author in [11].

(iii)  $\rightarrow$  (i) A. D. Weinstein's result above (Lemma 2.3) shows that, given a Riemannian  $G$ -strip  $P_G$ , one can glue it to a disk to obtain a surface  $S_G$ , and there exists a metric  $g$  on  $S_G$  agreeing with the original metric on  $P_G$ , and a point  $x$  in  $S_G$  with  $C(x) = G$ .  $\square$

We need one more result, well known in the graph theory.

**Lemma 2.5** *For every graph with  $m$  edges,  $n$  vertices, and  $q$  generating cycles holds  $q = m - n + 1$ .*

**Theorem 2.6** *Every metric graph can be realized as a cut locus.*

*Proof:* By Proposition 2.4, it suffices to provide, for every metric graph  $G$ , at least one  $G$ -strip.

We notice first that we can reduce our problem to the cyclic part  $G^{cp}$  of  $G$ . Assume  $G \setminus G^{cp}$  consists of finitely many finite trees, say  $T_1, T_2, \dots, T_m$ . Since every tree  $T$  has a “leaf”-type  $T$ -strip, one can attach (in a natural way) all the  $T_i$ -strips to a  $G^{cp}$ -strip to obtain a  $G$ -strip.

We proceed by induction over the number  $k$  of generating cycles of  $G$ .

For  $k = 0$  and  $G = G^{cp}$  the strip is elementary.

For  $k = 1$  and  $G = G^{cp}$  our strip is the flat compact Möbius band.

Assume now that there exist strips for all graphs with  $k$  generating cycles, for some  $k \geq 1$ .

Let  $G_{k+1} = G_{k+1}^{cp}$  be a metric graph with  $k + 1$  generating cycles, and  $e$  an edge of  $G_{k+1}$  in some generating cycle of  $G_{k+1}$ .

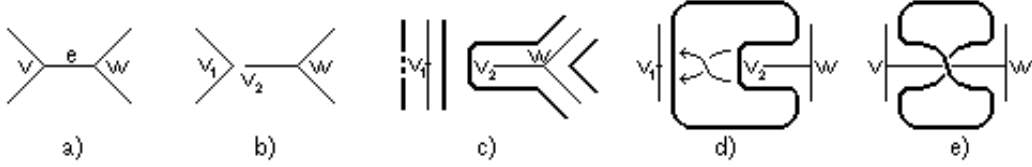


Figure 1: Induction reduction: edge  $e$  joins distinct vertices  $v \neq w$ .

Detach  $e$  from  $G_{k+1}$  at one extremity, say  $v$ ; Figure 1(a)-(b) presents the case when  $e$  joins distinct vertices  $v \neq w$ , while Figure 2(a)-(b) presents the case  $v = w$ . Denote by  $G_k$  the resulting metric graph, and by  $v_1, v_2$  the images of  $v$  in  $G_k$ .

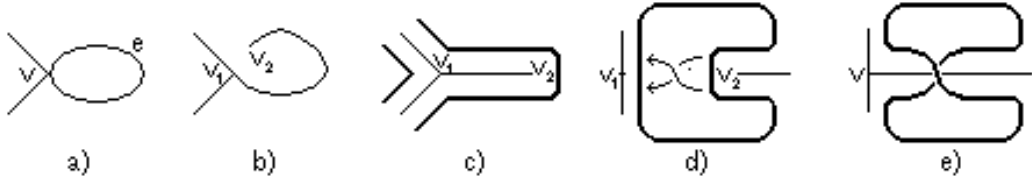


Figure 2: Induction reduction: edge  $e$  is a loop at  $v$ .

Since  $G_k$  has one vertex more than  $G_{k+1}$ , it has  $k$  generating cycles (see Lemma 2.5), and by the induction assumption there exists a  $G_k$ -strip  $P_{G_k}$  (see Figures 1(c) and 2(c)). Consider a planar representation of the boundary of  $P_{G_k}$  as a simple closed curve (illustrated in Figures 1(d) and 2(d)), and attach to it a switched  $e$ -strip, see Figures 1(e) and 2(e), to obtain a  $G_{k+1}$ -strip.  $\square$

**Remark 2.7** *Disconnected graphs can as well be realized as cut loci, but on non-complete surfaces. To see this, consider a disconnected graph  $G'$  as a subgraph of a connected graph  $G$ , and realize  $G$  as a cut locus on a surface  $S$ ; i.e.,  $G = C(x)$  for some point  $x$  in  $S$ . With  $T = G \setminus G'$ , we have  $C(x) = G'$  on  $S \setminus T$ .*

**Remark 2.8** *Our Theorem 2.6 shows, in particular, that for every connected graph  $G$ , there exists a 2-cell embedding with just one face, onto some surface  $S_G$ . This result is well known in the topological graph theory, see e.g. [23].*

**Question 2.9** *Several open questions naturally arise from Theorem 2.6.*

- i) Can the metric of the surfaces  $S_G$ , realizing  $G$  as a cut locus, be chosen analytic? See the result of S. B. Myers [24] mentioned in the introduction.*
- ii) Cut loci on Riemannian surfaces may be quite large sets, see the introduction. Can Theorem 2.6 be extended to infinite graphs?*
- iii) Can Theorem 2.6 be extended to higher dimensions?*

There usually are many strips on the same graph; we formalized this by several concepts [16], that we briefly present next.

**Definition 2.10** *A cut locus structure (shortly, a CL-structure) on the graph  $G$  is a strip on the cyclic part  $G^{\text{cp}}$  of  $G$ .*

**Definition 2.11** *Consider, for a point  $x$  on a riemannian surface  $(S, g)$  and for  $\varepsilon > 0$  small enough, the  $C(x)$ -strip obtained as the union, over all segments  $\gamma$  starting at  $x$  and parametrized by arclength, of the points  $\gamma(l_\gamma - \varepsilon)$ . We call the CL-structure constructed in this way the cut locus natural structure defined by  $x$ , and denote it by  $CLNS(x)$ , or by  $CLNS(x, g)$  if to point out (the dependence on) the metric  $g$ .*

With these notions, Can Theorem 2.6 can rephrased as that each graph possess at least one CL-structure, while Proposition 2.4 and Lemma 2.3 say that each CL-structure can be realized in a natural way.

In order to easier handle a CL-structure, we associate to it an object of combinatorial nature. To this goal, denote by  $\mathcal{P}$  and  $\mathcal{A}$  the set of all point-strips, respectively edge-strips, of a CL-structure  $\mathcal{C}$  on the graph  $G$ .

Denote by  $V$  the vertex set of  $G$ , and by  $E$  the edge set of  $G$ .

**Definition 2.12** *Consider an elementary decomposition of the  $G$ -strip  $P_G$  such that each elementary strip has a distinguished face, labeled  $\bar{0}$ . The face opposite to the distinguished face will be labeled  $\bar{1}$ . Here,  $\bar{0}$  and  $\bar{1}$  are the elements of the 2-element group  $(\mathbb{Z}_2, \oplus)$ .*

*To each pair  $(v, e) \in V \times E$  consisting of a vertex  $v$  and an edge  $e$  incident to  $v$ , we associate the  $\mathbb{Z}_2$ -sum  $\bar{s}(v, e)$  of the labels of the elementary strips  $\nu \in \mathcal{P}$ ,  $\varepsilon \in \mathcal{A}$  associated to  $v$  and  $e$ ; i.e.,  $\bar{s}(v, e) = \bar{0}$  if the distinguished faces of  $\nu$  and  $\varepsilon$  agree to each other, and  $\bar{1}$  otherwise. Therefore, to any cut locus structure  $\mathcal{C}$  we can associate a function  $s_{\mathcal{C}} : E \rightarrow \{\bar{0}, \bar{1}\}$ ,*

$$s_{\mathcal{C}}(e) = \bar{s}(v, e) \oplus \bar{s}(v', e), \quad (1)$$

*where  $v$  and  $v'$  are the vertices of the edge  $e \in E$ .*

*We call the function  $s_{\mathcal{C}}$  defined by (1) the companion function of  $\mathcal{C}$ .*

The value  $s_{\mathcal{C}}(e)$  above can be thought of as the switch of the edge  $e$ .

**Definition 2.13** *Assume first that the graph  $G$  is 2-connected. Two CL-structures  $\mathcal{C}, \mathcal{C}'$  on  $G$  are called equivalent if their companion functions are equivalent: i.e.,  $s_{\mathcal{C}}$  and  $s_{\mathcal{C}'}$  are equal, up to a simultaneous change of the distinguished face for all elementary strips in  $G$  (i.e., either  $s_{\mathcal{C}} = s_{\mathcal{C}'}$ , or  $s_{\mathcal{C}} = \bar{1} \oplus s_{\mathcal{C}'}$ ).*

*If  $G$  is not 2-connected, the CL-structures  $\mathcal{C}, \mathcal{C}'$  on  $G$  are called equivalent if their companion functions are equivalent on every 2-connected component of  $G$ .*

From now on, all CL-structures will be considered up to equivalence. This will allow us, whenever we consider surfaces realizing the graph  $G$  as a cut locus, to actually think about CL-structures and their companion functions on  $G$ .

The next two sections are related to the following.

**Question 2.14** *What can be said about the Riemannian surface  $S$  if, for every point  $x$  in  $S$ ,  $CLNS(x)$  does not depend on  $x$ ?*

### 3 Constant curvature realizations

In this short section we present a direct way to realize some graphs as cut loci, different from that provided by Theorem 2.6.

**Theorem 3.1** *Every CL-structure on a graph of constant order can be realized on a surface of constant curvature.*

*Proof:* Denote by  $G$  a cyclic graph of constant order  $k$ , and by  $\mathcal{C}$  a CL-structure on  $G$ .

If  $G$  is a point then the unique CL-structure on  $G$  can be realized as  $CLNS(x)$  for any point  $x$  on the unit 2-dimensional sphere.

Assume now that  $G$  is a cycle. Then again we have a unique CL-structure on  $G$ , and it can be realized as  $CLNS(x)$  for any point  $x$  on the standard projective plane.

Consider now a graph  $G$  with  $q \geq 2$  generating cycles; by Lemma 2.5, we get  $m \geq 2$ .

For  $m = 2$ , let  $F_{2m} = F_4$  denote the square in the Euclidean plane  $\Pi$ .

For  $m = 3$ , let  $F_{2m} = F_6$  denote the regular hexagon in  $\Pi$ .

For  $m \geq 4$ , consider a regular  $2m$ -gon  $F_{2m} = \bar{z}_1 \dots \bar{z}_{2m}$  in the hyperbolic plane  $\mathbb{H}^2$  of constant curvature  $-1$ , such that  $\angle \bar{z}_i \bar{z}_{i+1} \bar{z}_{i+2} = 2\pi/k$  (all indices are taken  $(\text{mod } 2m)$ ).

We view now the CL-structure  $\mathcal{C}$  on  $G$  as a closed path  $D$  in  $G$  containing all edges of  $G$  precisely twice, hence every vertex of  $G$  appears precisely  $k$  times in  $D$ .

We identify now the path  $D$  with (the boundary of)  $F_{2m}$ , such that each image in  $D$  of an edge of  $G$  corresponds to precisely one edge in  $F_{2m}$ , each image in  $D$  of a vertex of  $G$  corresponds to precisely one vertex in  $F_{2m}$ , and the order of edges and vertices along  $D$  is preserved. It remains to identify, for every edge  $e$  in  $G$ , its two images in  $F_{2m}$ , to obtain a differentiable surface  $S_G$  of constant curvature  $-1$ . By construction, the natural cut locus structure of the image  $x$  in  $S_G$  of the center of  $F_{2m}$  is precisely  $\mathcal{C}$ .  $\square$

**Remark 3.2** *With a similar proof, one can show that every CL-structure on an arbitrary graph can be realized on a surface of constant curvature with at most  $(n-p)$ -singular points (i.e., on an Alexandrov surface with curvature bounded below, see [30] for the definition). Here,  $p$  is the number of vertices in  $G$  of maximal degree.*



**Example 3.3** *The complete graphs  $K_r$ , the multipartite graphs  $K_{p_1, \dots, p_r}$ , as well as the graph of Petersen, can be realized as cut loci in constant curvature ( $r, p_1, \dots, p_r \in \mathbb{N}$ ).*

## 4 Stability

In this section we propose a notion of stability for cut locus structures, while in the next section we show that –roughly speaking– stability is a generic property of CL-structures. For our goal, we need to further investigate cut loci.

The cyclic part of the cut locus was introduced and first studied by J. Itoh and T. Zamfirescu [20].

**Proposition 4.1** *The cyclic part of the cut locus depends continuously on the point; i.e.,*

- (i) *if  $x_n \in S$ ,  $x_n \rightarrow x$ , and  $y_n \in C(x_n)$ ,  $y_n \rightarrow y$ , then  $y \in C(x)$ , and*
- (ii) *if  $x_n \in S$ ,  $x_n \rightarrow x$ , and  $y \in C(x)$ , then there exist  $y_n \in C(x_n)$  such that  $y_n \rightarrow y$ .*

*Proof:* The first part (i.e., the upper semi-continuity of cut loci, as the reference point varies on the Riemannian surface  $S$ ) is known and follows from the upper semi-continuity of geodesic segments.

For the second part, consider a sequence of points  $x_n \in S$ ,  $x_n \rightarrow x$ .

The number  $q$  of generating cycles in the cyclic part of a cut locus does not depend on the point in  $S$ , hence it is constant on  $S$ . Therefore,

$$q(C^{cp}(x_n)) = q(C^{cp}(x)).$$

Assume now that (ii) doesn't hold. Then there exist a point  $y \in C^{cp}(x)$  and a neighborhood  $N_y \subset S$  such that  $V_y \cap C^{cp}(x_n) = \emptyset$ , for  $n$  sufficiently large. Denote by  $C^-$  the set of such  $y$ s, and notice that  $C^-$  is an open subset of  $S$ .

Lemma 2.5 and (i) show now that

$$\lim_n q(C^{cp}(x_n)) = q(\lim_n C^{cp}(x_n)) = q(C^{cp}(x) \setminus C^-) < q(C^{cp}(x)),$$

and a contradiction is obtained. □

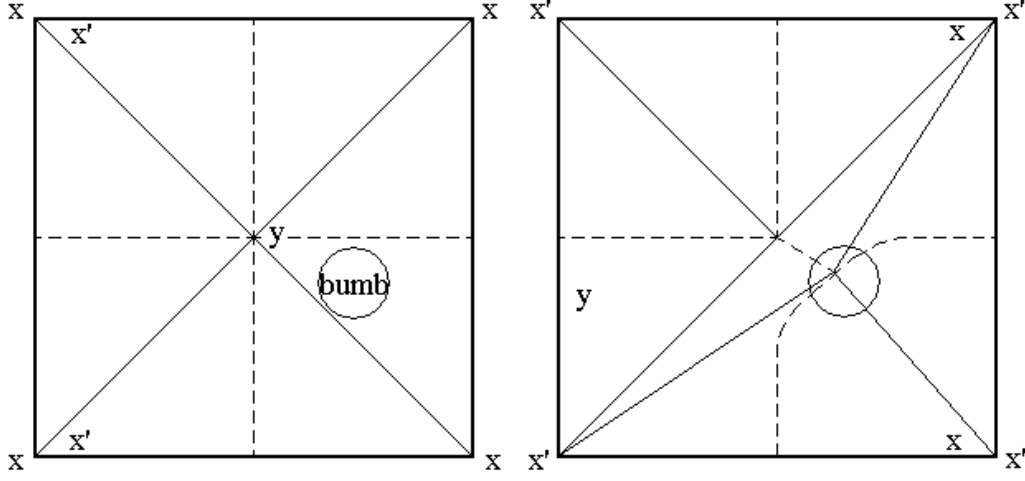


Figure 3: Unstable cut locus structure.

**Definition 4.2** Consider a CL-structure  $\mathcal{C}$  on the graph  $G$ , a Riemannian surface  $(S, g)$  and a point  $x \in S$ .  $\mathcal{C}$  is called stable with respect to  $x$  in  $S$  if  
(i)  $CLNS(x) = \mathcal{C}$ , and  
(ii) there exists a neighborhood of  $x$  in  $S$ , for all points  $y$  of which holds  $CLNS(y) = \mathcal{C}$ .

**Definition 4.3** The CL-structure  $\mathcal{C}$  is called globally stable if it is stable on all surfaces where it can be realized as a CLNS.

**Remark 4.4** Assume we have distinct pairs  $(S, x)$  and  $(S', x')$  of Riemannian surfaces  $S, S'$  and points  $x \in S, x' \in S'$  such that  $CLNS(x) = CLNS(x') = \mathcal{C}$ . If  $\mathcal{C}$  is stable with respect to  $(S, x)$ , it is not necessarily stable with respect to  $(S', x')$ , as the following example shows.

**Example 4.5** i) Any CL-structure on a graph of constant order  $k > 3$  is stable with respect to the natural realization given by Theorem 3.1.

ii) We roughly explain here how to produce unstable CL-structures from those stable CL-structures at (i).

Consider, for example, a square fundamental domain of a flat torus  $T$  with a bump, see Figure 3 left. The cut locus of the point  $x \in T$  represented at the corners of the square, is the 4-graph with one vertex  $y$ , as indicated by the dashed line. The four segments from  $x$  to  $y$  are also indicated, and are

not affected by the bump. We choose  $x$  such that one segment is tangent to the bump's boundary.

Now consider a point  $x'$  arbitrarily close to  $x$ : slightly move  $x$  to “the right”, for example, to  $x'$ , see Figure 3 right. There remain three segments from  $x'$  to  $y$ , those in the upper-left half-domain; they are all shorter than the geodesic joining  $x'$  to  $y$  that crosses the bump, so  $y$  is a vertex of degree three in  $C(x')$ . There is another vertex of degree three in  $C(x')$ , also indicated in the figure together with the segments joining it to  $x'$ . In this case,  $C(x')$  is a 3-graph with two vertices and two generating cycles. J. Itoh and T. Sakai describe into details a similar procedure, see Remark 2.7 in [15].

In conclusion, the 4-graph with one vertex is not stable with respect to  $x$  in  $T$ .

**Theorem 4.6** *A cut locus structure on the graph  $G$  is globally stable if and only if  $G$  is a 3-graph.*

*Proof:* Let  $\mathcal{C}$  be a locus structure on  $G$ .

Assume first that  $G$  is a 3-graph; then its cyclic part is itself a 3-graph. Assume, moreover, that  $\mathcal{C}$  is realized as  $\mathcal{C} = CLNS(x)$ , for some point  $x$  on some Riemannian surface  $S$ .

Now, for points  $x_n \in S$ ,  $x_n \rightarrow x$ , Proposition 4.1 gives  $\lim_n C^{cp}(x_n) = C^{cp}(x)$ .

Assume that, for our sequence  $\{x_n\}$ , we have vertices  $z_n$  in  $C^{cp}(x_n)$  of degree  $d$  larger than 3, say  $d = 4$  (the case  $d > 4$  is similar).

Denote by  $B_n^i$  the branches of  $C^{cp}(x_n)$  incident to  $z_n$ ; there exist segments  $\gamma_n^i, \gamma_n^{'i}$  from  $x_n$  to  $z_n$  ( $i = 1, \dots, 4$ ) and a neighborhood  $V_n$  of  $z_n$  in  $S$ , such that one of the sectors around  $z_n$  determined by  $\gamma_n^i, \gamma_n^{'i}$  and  $V_n$  contains  $B_n^i \cap V_n$  but no other subsegment of a segment from  $x_n$  to  $z_n$ .

Take some limit point  $z$  of  $z_n$ ; then  $z \in C^{cp}(x)$ , because  $\lim_n C^{cp}(x_n) = C^{cp}(x)$ , and  $z$  has degree 3 in  $C^{cp}(x)$ , by our assumption that  $C^{cp}(x)$  is a cubic graph. Therefore, there exists  $1 \leq i \leq 4$  such that the segments  $\gamma_n^i$  and  $\gamma_n^{'i}$  have a common limit  $\gamma^i$ , which is a segment from  $x$  to  $z$ . Then, for  $n$  large enough,  $\gamma_n^i \cup \gamma_n^{'i}$  bounds a region of  $S$  contractible to a point and intersecting  $B_n^i \cap V_n \setminus \{z_n\}$ . Since  $C^{cp}(x_n)$  intersects  $\gamma_n^i \cup \gamma_n^{'i}$  only at  $z_n$ , it follows that  $C^{cp}(x_n)$  contains a tree with the root at  $z_n$ , and a contradiction is obtained.

Concluding, the graph  $C^{cp}(x_n)$  has to be cubic, and now  $\lim_n C^{cp}(x_n) = C^{cp}(x)$  implies that the cyclic parts of  $C(x)$  and  $C(x_n)$  are homeomorphic, and we thus  $G$  is stable.

Assume now that  $G$  is stable and it has a vertex  $y$  of degree strictly larger than 3, and consider a point  $x$  in the Riemannian surface  $S$  such that  $\mathcal{C} = CLNS(x)$ . Then, by “putting” a bump tangent to one of the segment from  $x$  to  $y$  (i.e., modifying the metric on  $S$  accordingly) we obtain a new metric on  $S$  with respect to which we still have  $\mathcal{C} = CLNS(x)$ , but we have points  $x'$  arbitrarily close to  $x$  such that  $CLNS(x') \neq \mathcal{C}$ , see Example 4.5 or Theorem 5.2.  $\square$

The following is, in some sense, opposite to Question 2.14.

**Question 4.7** *How many stable CL-structures can exist on a given surface?*

Upper bounds on the number of cut locus structures on a graph are obtained in [19].

## 5 Generic behavior

We shall make use of the main result in [6], given in the following as a lemma. For, denote by  $\mathcal{G}$  the space of all Riemannian metrics on the surface  $S$ ; i.e., it is viewed as the space of sections of the bundle of positive definite symmetric matrices over  $S$ , endowed with the  $C^\infty$  Whitney topology [6].

Recall that a metric  $g$  on the surface  $S$  is called *cut locus stable* [6] if for any  $h$  close to  $g$  there is a diffeomorphism  $\phi$  of the surface, depending continuously on  $h$ , such that  $\phi(C(x, g)) = C(x, h)$ ; here,  $C(x, g)$  denotes the cut locus of  $x$  with respect to  $g$ .

**Lemma 5.1** [6] *For every point  $x$  in  $S$  there exists a set  $\mathcal{B}_x$  of  $C(x)$  stable metrics on  $S$ , open and dense in  $\mathcal{G}$ . Moreover, for any  $g$  in  $\mathcal{B}_x$ , every ramification point of the cut locus of  $x$  with respect to  $g$  is joined to  $x$  by precisely three segments.*

In virtue of Definition 2.13 and the remark following it, we can regard a CL-structure on the graph  $G$  as a function  $G \rightarrow Z_2$ .

**Theorem 5.2** *There exists an open and dense set in  $S \times \mathcal{G}$ , for every element  $(x, g)$  of which the naturally defined cut locus structure  $CLNS(x, g)$  is cubic and locally constant.*

*Proof:* Consider the subset  $\mathcal{O}$  of  $S \times \mathcal{G}$ , containing all pairs  $(x, g)$  for which the naturally defined cut locus structure  $CLNS(x, g)$  is cubic.

The density of  $\mathcal{O}$  in  $S \times \mathcal{G}$  follows directly from Lemma 5.1.

We show next the openness of  $\mathcal{O}$  in  $S \times \mathcal{G}$ . Assume this is not the case, hence there exist  $(x, g) \in \mathcal{O}$  and a sequence  $\{(x_n, g_n)\}_n \subset S \times \mathcal{G}$  convergent to  $(x, g)$ , such that  $C^{cp}(x, g)$  is a cubic graph, but  $C^{cp}(x_n, g_n)$  are not cubic.

For  $n$  sufficiently large,  $C^{cp}(x, g_n)$  are still cubic graphs, by Lemma 5.1. Moreover, an argument similar to the first part in the proof of Theorem 4.6 shows now that, for  $g_n$  close enough to  $g$ ,  $C^{cp}(x, g_n)$  is a cubic graph homeomorphic to  $C^{cp}(x, g)$ .

Now, Theorem 4.6 shows that  $C^{cp}(x_n, g_n)$  is a cubic graph homeomorphic to  $C^{cp}(x, g_n)$ , hence homeomorphic to  $C^{cp}(x, g)$ , and a contradiction is obtained.

Therefore,  $\mathcal{O}$  is open in  $S \times \mathcal{G}$  and, moreover, for every pair  $(x, g)$  in  $\mathcal{O}$  the naturally defined cut locus structure  $CLNS(x, g)$  is locally constant.  $\square$

The following result is well-known.

**Lemma 5.3** *Every graph can be obtained from some cubic graph by edge contractions.*

**Remark 5.4** *Passing from a stable CL-structure to another stable CL-structure is realized via a non-stable CL-structure, one that –in particular– lives on a non-cubic graph (see Theorem 5.2 4.6 and Lemma 5.3). This is realized in the first step by contracting one or several edge-strip(s), and in the second step by an operation, that we can think about as “blowing up” all vertices of degree larger than 3 to trees of order 3. (A formal description of this is given in [17].)*

**Remark 5.5** *Non-isometric surfaces realizing the same graph  $G$  as a cut locus (Theorem 2.6) are homeomorphic to each other, since topologically they can be distinguished only by their genus, which is a function on the number of generating cycles of  $G$ . Therefore, all distinct CL-structures on  $G$  “live” on homeomorphic surfaces. On the other hand, Theorem 5.2 shows in particular that “equivalent” CL-structures on  $G$  (the precise definition is given in [17]) can be realized on non-isometric surfaces.*

**Acknowledgement** This work was begun during C. Vîlcu’s stay at Kumamoto University, supported by JSPS; its completion at IMAR Bucharest was partially supported by the grant PN II Idei 1187.

The authors are indebted to all persons who contributed with remarks and suggestions to the present form of the paper.

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